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Oscillations in Clarinet-like Systems

A Statis Report

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This is a brief preliminary report of work which is underway. The given references are not meant to be taken as being complete. Certain statements are made in this paper which are not justified as fully as possible. It is hoped that enough has been included to make them at least plausible.

A wind instrument is a complex, non-linear oscillator. Its source of energy is, in the final analysis, the difference between the air pressure in the player's mouth and that in the surrounding air. The player does work to compress the air; and, when the air is allowed to expand in going through the system, the energy lost by it leaves the system as heat and as radiation of sound waves. Only a small amount of energy goes into the production of external sound waves.

For steady state oscillations the energy source is the very nearly constant pressure in the player's mouth. The pressure is not strictly constant as the motion of the reed or Lips will affect the air in the mouth causing fluctuations related to those insdie the mouthpiece. We will show shortly that in the case of woodwinds this coupling is small enough to be negligible for present purposes.

In a steady state situation the system is observed to oscillate with a definite frequency. The energy source is a constant pressure, and the frequency of oscillation is dictated by the oscillating system itself. The frequency at which an instrument plays is determined by the nature of its various parts which may include things like air (which may vary in composition), the reed or lips, the bore, the embouchure, the pressure in the mouth, the parometric pressure, and the temperature.

We will deal specifically with reed woodwinds according to the following model. The pressure in the player's mouth is represented by a constant pressure or at least by some pressure which is fixed by the player and does not depend at all on what the oscillator is doing. This model, in effect, assumes that the mouth is a perfect constant pressure source with zero output impedance. As remarked earlier, this is an approximation, since the mouth cavity and vocal tract are coupled to the oscillating air in the instrument. That the effect is small can be seen by considering the variety of people who can play the same instrument on pitch with a tone that is recognizable as belonging to the individual instrument.

Figure 1 shows a stylized version of a clarinet. Air enters the instrument from the player's mouth - the constant pressure source. It flows into the instrument through an opening which is controlled in some manner by the movement of the reed. We shall call the <u>volume flow</u> of air into the instrument from the mouth $U(t)$. This air flow splits up inside the mouthpiece and goes two places. Part of it goes down the bore and part goes to moving the reed itself. The bore can be characterized by an input impedance Z_b, defined as the ratio of mouthpiece pressure to volume flow. The resistance of the reed to being moved (represented in the diagram by the plunger on a spring) may be characterized by an impedance Z_t, so called because the reed forms an elastic termination at the end of the bore. The two impedances are fed in parallel by the flow of air in the neighborhood so that the net impedance $Z(\omega)$ (a function of frequency) seen at the player's air inlet to the mouthpiece is

$$Z(\omega) = \frac{Z_t Z_b}{Z_t + Z_b} \quad .$$

The impedance of the bore Z_b may be calculated, or it can be found by measuring the input impedance of the bore at the mouthpiece end keeping the reed clamped rigidly shut. This is a standard measurement. We describe later how Z_t is obtained from experimental data. It is worthwhile however, to have a general idea of what $Z(\omega)$ looks like. In the normal woodwind the net input impedance $Z(\omega)$ has two or more peaks which are approximately integrally related in frequency. The Q of these peaks is, in the case of the clarinet, about 20. In fact $Z(\omega)$ looks very much like the impedance Z_b of the bore alone, but the peaks are slightly displaced. The reed impedance Z_t acts as a perturbation which moves the impedance peaks down in frequency. As we will show, the instrument plays at a frequency very near the top of the $Z(\omega)$ peak.

The impedance $Z(\omega)$ has been defined already as the ratio of the pressure in the mouthpiece P'(t) the volume flow of air coming into the mouthpiece u(t).

$$Z(\omega)\frac{p'(t)}{u(t)} \tag{2}$$

This equation has only formal significance, however because p'(t) and n(t) in general contain components of many different frequencies whereas $Z(\omega)$ is defined at a single frequency . The problem of giving a better definition will be dealt with later in this report.

The next assumption is that the flow through the reed can be expressed as a function of the reed position y by means of a Taylor's expansion in powers of y. At first sight it would appear improper to assume that the flow is dependant only on the size of the opening and not on the pressure difference across it.*

We designate the position of the reed by its displacement, y from its equilibrium position. This displacement can be calculated by treating the reed as a driven harmonic oscillator. The statement that the flow can be written as a Taylor's series in the reed position is made mathematically explicit in Eq. 3.

$$u(t) = \sum_{k=0}^{\infty} A_k [y(t)]^k. \tag{3}$$

The A_k are a set of constants that are defined for a given reed, mouthpiece, and playing conditions. There is considerable experimental evidence which leads one to believe that the series converges rapidly. We will cut the series off at the k=N term where N is less than about 4. We have

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*(For instance John Backus states that the steady flow u for a clarinet reed is given quite well by $u = 37\, p^{2/3}\, y^{4/3}$ where p is the pressure difference across the opening and y is the separation between the tip of the reed and the tip of the mouthpiece. JASA March 1963, No. 35, No. 3 p. 305).

$$u(t) = \frac{p'(t)}{Z(\omega)} = \sum_{k=0}^{\infty} A_k [y(t)]^k. \tag{4}$$

Having made the assumption that the reed behaves like a driven, damped, harmonic oscillator, its characteristics can be expressed in terms of the reed natural frequency ω_r, the half-power bandwidth g_r, the effective mass μ_r, and an area S_r which is the effective area which the driving pressure works on. (S_r is represented as the area of the piston in Figure 1).

The equation of motion of the reed is

$$\frac{d^2y}{dt^2} + g_r\frac{dy}{dt} + \omega_r^2 y = \frac{p(t)S_r}{\mu_r} \tag{5}$$

where p(t) is the pressure difference across the reed. If the reed is driven by a simple harmonic driving force $p(t) = p_o e^{i\omega t}$, we get a solution

$$y = \frac{p_o S_r e^{i\omega t}}{\mu_r[(\omega_r^2-\omega^2)+ i\omega g_r]} = D(\omega)p(t), \tag{6}$$

where this equation defines the function $D(\omega)$. The impedance that the reed shows to the driving pressure is the pressure p divided by the volume flow u(t). Here

$$u(t) = S_r \frac{dy}{dt} = i\omega S_r y(t)$$

so that

$$Z_t(\omega) = \frac{\mu_r}{S_r^2} \left(\frac{\omega_r^2-\omega^2}{i\omega} + g_r \right) \tag{7a}$$

and

$$D(\omega) = \frac{S_r}{\mu_r[(\omega_r^2-\omega^2)+i\omega g_r]} \tag{7b}$$

The connection between y and p for a given frequency component is given by the function $D(\omega)$. We can replace y by $D(\omega)$ p in equation (4) where $D(\omega)$ is now written in the same formal sense that $Z(\omega)$ is

$$\frac{p'(t)}{Z(\omega)} \sum_{k=0}^{N} A_k [D(\omega)p(t)]^k. \tag{8}$$

Here p(t) is the oscillatory time dependent part of the pressure in the mouthpiece, and p'(t) is the net pressure in the mouthpiece; it is related to p(t) by Eq. 9.

$$p'(t) = p(t) + \bar{p} \qquad (9)$$

$\bar{p}$ is the average pressure in the mouthpiece. This is measured with respect to the pressure of the outside air and is quite small. We have

$$\frac{\bar{p}+p(t)}{Z(\omega)} = \sum_{k=0}^{N} A_k[D(\omega)p(t)]^k \quad . \qquad (10)$$

The general features of the equation and its solutions are discussed in a paper entitled Sound Production in Wind Instruments by A.H. Benade and D.J. Gans. (Annals of the New York Academy of Sciences Vol. 155 article 1 p. 247-263 Nov. 20, 1968).

We must now say more about $Z(\omega)$ and $D(\omega)$. Expressions for $Z_t(\omega)$ and $D(\omega)$ are given in Equs. 7. Note that they are complex numbers and therefore involve phase shifts. They are expressed in terms of reed characteristics which must be found experimentally. The specific work done so far has been done with clarinets because the single reed and cylindrical bore makes them easier to work with than the other reed woodwinds. Experiments for getting the reed characteristics for a clarinet reed and mouthpiece will now be discussed.

The natural frequency of the reed can be found quite simply. One has only to attach the output of a microphone to the input of an oscilloscope and, holding the reed near the microphone, tap it sharply at about the center of the scraped part of the reed. The proper place can be found easily with a little practice. The trace on the scope shows that the reed moves as a damped harmonic oscillator. The natural frequency ω_r can be read directly from a photograph of the scope trace. The reed we used had a natural frequency of about 1550 Hz when wet. One cannot get the relevant bandwidth g_r from the photograph as the lips add a great deal of damping to the reed under playing conditions.

One can get the two remaining numbers for the reed by computing the flattening of the played frequency of a clarinet due to the reed impedance Z_t. That is, one measures the natural resonant frequencies of the pipe (peaks of $Z_b(\omega)$) and then the frequencies it actually oscillates at when played. These frequencies will be essentially at the tops of the $Z(\omega)$ peaks. The expected movement of the peaks from Z_b to $Z(\omega)$ can be calculated from Equs. 4 and 7, and the reed characteristics can be deduced from the observed flattening.

The observed flattening for the chalumeau register "open G" was found to be about 42¢. If one changes to the second register with out changing embouchure the twelfth is found to be about 20¢ flat. Therefore the net flattening due to the reed at a frequency equal to the 3rd harmonic of the chalumeau "open G" is about 20 + 42 = 62¢.

The reed characteristics calculated from these shifts and from ω_r are then

$$g_r = 1.33 \times 10^4 \mathrm{sec}^{-1} \qquad \text{and}$$

$$\frac{\mu_r}{S_r^2} = 1.36 \times 10^{-2} \mathrm{gm/cm}^4 \quad .$$

If we define $\sigma = \frac{S_r}{S}$ as the ratio of the effective area S_r to the bore cross sectional area $S(\sim 1.77\mathrm{cm}^2$ of a clarinet, we get

$$\frac{\mu_r}{\sigma^2} = 4.36 \times 10^{-3} \mathrm{gm}.$$

Both S_r and S are shown on Fig. 1. For a clarinet σ is about unity and we will take it to exactly so. The observed Q of the reed unoer playing conditions is about .73 (Q is the ratio of ω_r to g_r). Knowing these reed characteristics completely defines $D(\omega)$ and $Z_t(\omega)$. Since $Z_b(\omega)$ can be measured and $Z(\omega)$ claculated from Eq. 4, $D(\omega)$ and $Z(\omega)$, are now known functions.

It is found from the reed characteristics that $D(\omega)$ is a very flat function of frequency out past the 3rd harmonic of the chalumeau note being considered. (2358Hz), The magnitude of $D(\omega)$ at 716Hz is .99 of the value at 358Hz and at the 3rd harmonic (1074Hz). It is down to .92 of the value at 358Hz. Since the response curve of the reed is quite flat and the strongest spectral in the oscillation is the fundamental, it is not unreasonable to take the reed displacement to be approximately proportional to the pressure, and in phase with it. The power series for u can then be interpreted as the product of a power series in the pressure and one in the displacement of the reed. We will in fact use the <u>displacement</u> rather than the <u>pressure</u> in the expansion. The Λ_k must now <u>be interpreted</u> as constants <u>which have</u> no straight forward physical meaning. The work to date has been largely concerned with trying to deduce something about the coefficients.

We will now return to a more mathematical discussion of Eq. 10. We begin by specializing it to the case of steady state oscillations. If a steady state oscillation is taking place in a musical instrument the pressure in the mouthpiece must be a periodic function of time. We can therefore express it in terms of a Fourier series. That is, we can write

$$p(t) = \sum_{n=1}^{\infty} p_n \cos(n\omega t + \phi_n), \qquad (11)$$

recalling that p(t) was the time dependant part of the pressure in the mouthpiece. The ϕ_n are phase angles and we are at liberty to take $\phi_1 = 0$ as this corresponds to choosing an origin for the time. The $Z(\omega)$ and $D(\omega)$ will multiply each Fourier component of the pressure by a constant appropriate to its frequency and also shift its phase. We have

$$\frac{\bar{p}}{Z_o} + \sum_{n=1}^{\infty} \frac{p_n}{Z_n} \cos(n\omega t + g_n) = \sum_{k=0} A_k \{ \sum_{n=1}^{\infty} p_n e_n \cos(n\omega t + \phi_n)\}^k \qquad (12)$$

where the Z_n and g_n correspond to the $\frac{1}{Z(\omega)}$ and the e_n and ϕ_n correspond to the $D(\omega)$.

We start with the linear case of this equation (N=1) as the simplest meaningful case. The equation becomes

$$\frac{\bar{p}}{Z_o} + \sum_{n=1}^{\infty} \frac{p_n}{Z_n} \cos(n\omega t + g_n) = A_o + A_1 \sum_{n=1}^{\infty} p_n e_n \cos(n\omega t + \phi_n) \qquad (13)$$

Since this equation must be true at all times it imposes a separate condition on each frequency component. These conditions are

$$\frac{\bar{p}_o}{Z_o} = A_o, \quad \text{and} \qquad (14)$$

$$\cos(n\omega t + g_n) = A_1 e_n Z_n \cos(n\omega t + \phi_n), \; n = 1, 2, \ldots \qquad (15)$$

Equation 14 is an equation for the time independant flow. It says that A_o is equal to the average volume flow of the air out of the system. This average volume flow is the drift velocity of the air entering the bore times the cross sectional area of the bore.

Equation 15 are an infinite number of equations - one for each harmonic. There are two solutions to each. One could have either $A_1 e_n Z_n = 1$ with $g_n = \phi_n$ or $A_1 e_n Z_n = -1$ with $g_n = \phi_n + \pi$. It is resonable that A_1 should be positive as this corresponds to more flow when the reed opening is larger. Since e_n and Z_n are also positive the solution of interest is

$$A_1 e_n Z_n = 1 \qquad g_n = \phi_n. \qquad (16)$$

The interpretation of this is as follows. A steady state oscillation can exist at a certain harmonic only if $A_1 e_n Z_n = 1$. Also the system will run off resonance enough to satisfy the requirement $g_n = \phi_n$. Note that this is an unstable situation. If $A_1 e_n Z_n$ becomes slightly greater than one the oscillation at that frequency will increase without limit. If $A_1 e_n Z_n$ becomes less than one the oscillation at that frequency will die out. However if $A_1 e_n Z_n = 1$ the oscillation will be steady and can have <u>any amplitude</u>. This is a characteristic of linear systems.

Figure 2 shows the conditions imposed on a linear oscillator. The top graph shows the tangents of the phase angles as a function of frequency. The requirement on phases is that the system can only oscillate at frequencies where the curves intersect. The lower graph shows the impedance Z as a function of frequency and also a threshold line $\frac{1}{A_1 e}$ as a function of frequency. The system can maintain oscillation only when the impedance is at or above this threshold. These regions are marked with vertical dotted lines. It is seen that for the system shown only two oscillation frequencies are possible. Since Z $\frac{1}{A_1 e}$ for both these frequencies if an oscillation starts it will increase exponentially until the system becomes non-linear, so that the amplitude saturates, or until the system is destroyed. Note that the impedance peaks are shown to get smaller in magnitude as the frequency increases. This is generally the case because of the nature of radiation and boundary layer losses. The threshold curve also rises as e decreases with frequency. Both these phenomena have the effect of making low frequency oscillation more favorable than high frequency oscillation. The original version of Figure 2 was prepared for a paper given by A.H. Benade at the National Clarinet Clinic in Denver in the summer of 1967.

We can identify $A_1 e_1 Z_1$ as the open loop signal gain at the frequency <u>for a linear system</u> considered as a feedback device. This is an important parameter which is a measure of how easily a system can oscillate and we shall give it the name η_1 for reasons which will become clear later. We shall be able to find η_1 for non-linear systems and interpret the resulting value in terms of the theoretical linear case.

We now turn to the non-linear case, N (in Eq. 4) being now taken as 3. It was thought advisable to have some simple clarinet-like systems to work with. To this end a series of 3 clarinet-like pipes was construct for which the function $Z(\omega)$ has 1,2, and 3 strong resonance peaks. Each pipe used a series of evenly spaced open holes at its lower end to act as a high pass filter, so that there were no strong resonances above the cutoff frequency.*

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*"On the Mathematical Theory of Woodwind Finger Holes", A.H. Benade. <u>JASA</u> Vol. 32, No. 12, 1591-1608, December, 1960.

These tubes were designed to play with the same frequency as a chalumeau "open G" of a clarinet. The 2-resonance pipe was made to approximate the $Z(\omega)$ of a Boehm clarinet. Figure 3 shows the impedance curve Z_b of the 1 and 3 resonance pipes and Figure 4 shows those for the 2 resonance pipe and for the open G of a good Boehm clarinet. Recall that function $Z(\omega)$ is very similar to Z_b and the main difference is that the peaks are shifted slightly in frequency.

We will consider the case of the pipe with one resonance peak. The pressure p(t) (inside the mouthpiece) was measured with a probe microphone while the tube was being blown. Ih the calculation discusses below the p(t) is taken as a known quantity and values of the A_z calculated.

It is convenient to start by writing Eqs. 12 for N=3 in dimensionaless, normalized form. This is done by making the following definitions:

$$p_n = k_n p_1,\ e_n = \varepsilon_n e_1,\ Z_n = \frac{\eta_n}{A_1 e_1},\ \text{and}\ \left(\frac{p_1 e_1}{2}\right)^{k-1} A_k = \alpha_k A_1\ . \tag{17}$$

These define k_n, ε_n, η_n, and α_k. Note that k_1=1, ε_1=1, and α_1 = 1. We have $\eta_1 = A_1 e_1 Z_1$ as before. With these definitions Eq. 12 becomes

$$\frac{k_o}{\eta_o} + \sum_n \frac{k_n}{n} \cos(n\omega t + g_n + Q_n) = \sum_{k=0}^{3} 2^{k-1} \alpha_k \{\sum_n k_n\ {}_n \cos(n\omega t + \delta_n + \phi_n)\} \tag{18}$$

All quantities in the equation are now dimensionalless including **nωt.**

This equation is much more difficult to deal with than the linear case, because the non-linearity adds restrictions on the results. For example we can now expect amplitude stability. A steady state solution also implies definite values for each p_n. Because the non-linearity couples the oscillations at different frequencies together we can expect a definite tone color and that the frequency of oscillation and ease of oscillation will depend on the shapes of Z(ω) and D(ω) not only near the playing frequency, but also near harmonics of the playing frequency. This can give a player a woodwind more or less control over the amount he can vary the pitch of a given not.

To get results it was assumed that the phase shifts ϕ_n, δ_n, and g_n all vanish. Thi is not as bad an assumption as it seem at first. For the 1 resonance pipe only the fundamental and 2nd and 3rd are present to any extent in the observed spectrum. In fact the relative amplitudes are $k_1 = 1$; $k_2 = 0.24$, $k_3 = .042$. There is strong indication from the shape of the wave form as displayed on an oscilloscope (with most of the fundamental filtered out) that the 3rd harmonic is 180° out of phase with the fundamental. This can be taken care of by using k_3 as a negative number in the equations and $\phi_3 = 0$. ($\phi_1 = 0$ by so defining the origin of time). K_2 and ϕ_2 are small enough so that it will not disrupt things to take $\phi_2 = 0$. Taking δ_n and g_n is not as strong a condition as it first seems. These quantities appear as $\delta_n - g_n$ in all the large terms in the equations. Taking $\delta_n = 0$ and $g_n = 0$ is essentially the same condition as $\delta_n = g_n$ **which** is the condition required for the phases in the linear case. The assumption $g_n = 0$, $\delta_n = 0$ is therefore essentially the condition that the presence of the non-linearity not change the playing frequency much. This is well known to be true.

The experimental values of the k_n were used and the values of the α's and η_1 found by two different approximate methods one derived from Lord Rayleigh.[1] The results of these two approximations are

1. $\alpha_2 = .75$, $\alpha_3 = -.25$, $\eta_1 = 2.72$ and

2. $\alpha_2 = .71$, $\alpha_3 = -.23$, $\eta_1 = 2.64$.

It was found that a solution is possible only if k_3 is taken to be negative. This, as mentioned before, is an agreement with the experiment. Note that α_3 is negative. There are definite indications that it should be. The values of the α's and η_1 are of a very reasonable size which is encouraging. The $\eta_1 = 2.7$ value is interpreted to mean that the oscillation is not as favored with the non-linearity as it would be if the system were linear. This value of η_1 implies that with the same reed it takes at most 3 times as much impedance to make the non-linear oscillator run as to make the linear one run. Another way of saying it is that for the given bore it takes a reed with an linear open loop signal gain of 2.7 to make the system maintain its oscillation. The linear oscillation would involve only the fundamental in this case. The higher frequency oscillations would be left to their own devices and would quickly decay away to zero as the impedance is high only at the fundametal. The non-linearity demands that these components by present, however. The high frequency oscillations must be maintained by energy coming from the oscillation at the fundamental. This understandably acts as a drag on the system that must be compensated for by a more effective reed.*

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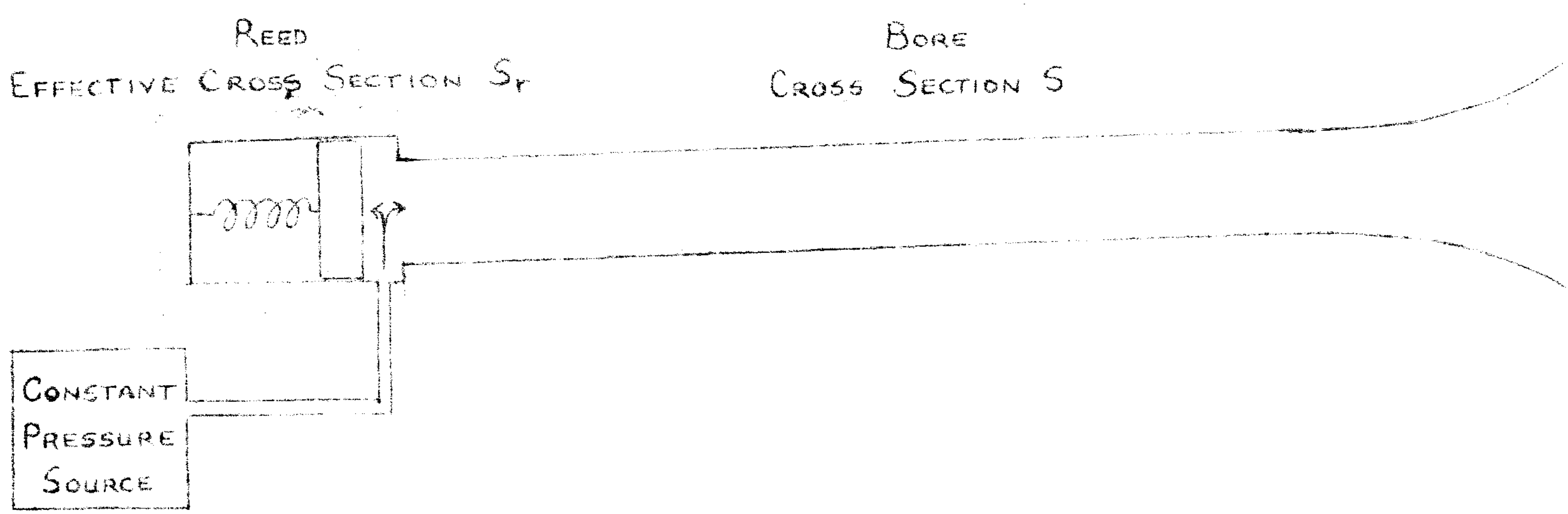
*A net amount of energy enters the system at the frequencies where $A_1 e_n Z_n > 1$ and a net amount leaves at others. A balences must be reached.

The system in practice demenstrates this reluctance to oscillate very well. It is impossible to play loudly on it and if one is out of practice or the reed is a bit dry it is a fight to get it started at all.

In the cases where there is more than one resonance the impedance is high at one or more harmonics. Energy can therefore also enter the system through oscillations at these frequencies which will make it easier for the system to sustain oscillation.

Projected Work

We hope to do more thorough work on the 2 and 3 resonance pipes soon. These are more difficult to work with because the approximate methods used for solution do not lead to as rapid convergence in these cases. The approximations also do not hold quite as well. We should soon be able to measure the phase angles ϕ_n which should help considerably. The possibility of using a product of two power series for u(t) instead of a single power series will also be explored. We also hope to look into the non-steady-state aspects of the problem which include such questions as how the frequency of oscillation is determined and what can be said about attack and transitions from one note to another. It would also be instructive to try the formalism on double reed instruments, but this could be a long time coming.



STYLIZED VERSION OF A CLARINET

Figure 1

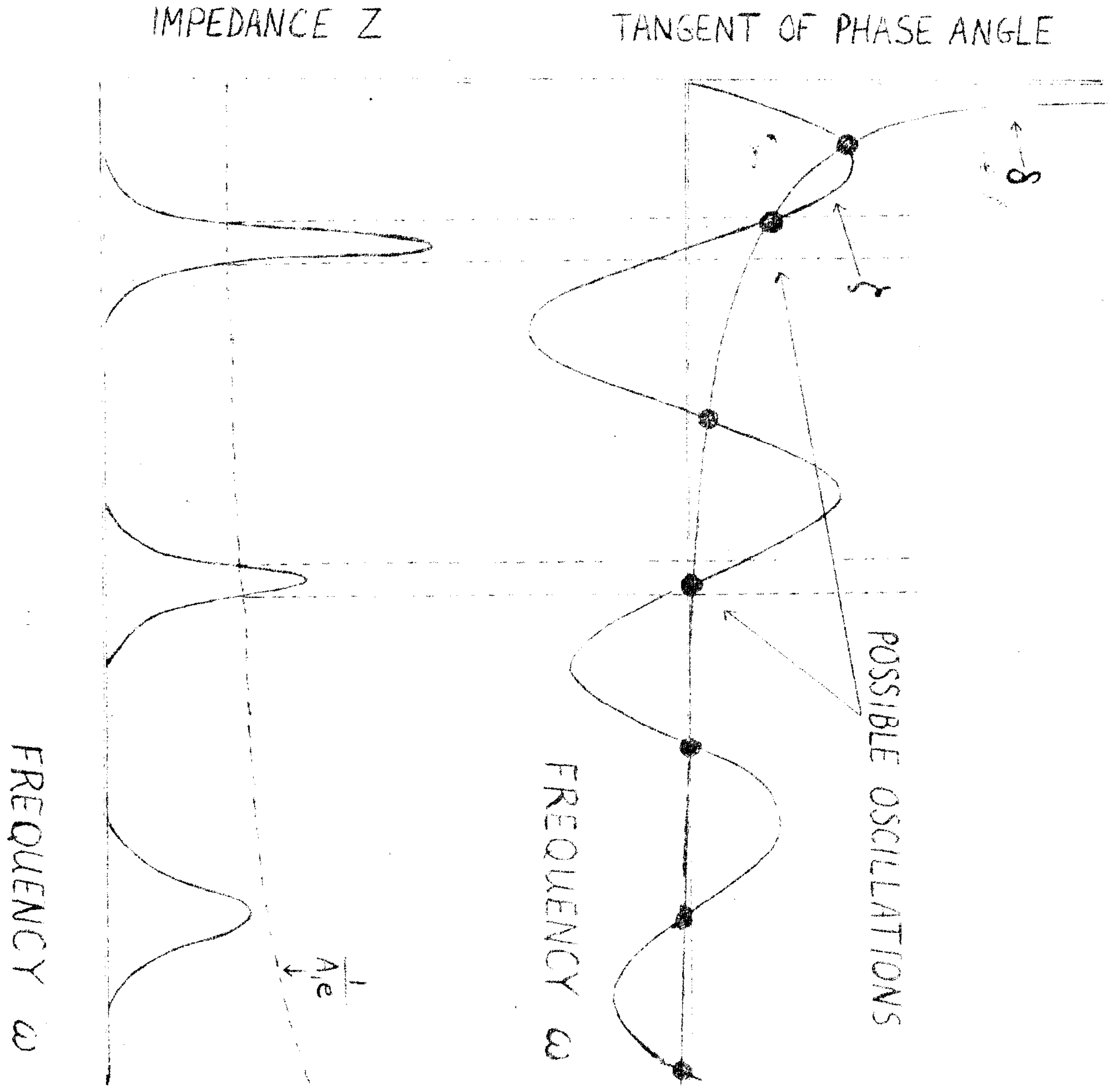

IMPEDANCE Z
TANGENT OF PHASE ANGLE
δ
γ
POSSIBLE OSCILLATIONS
FREQUENCY ω
FREQUENCY ω

FIGURE 2

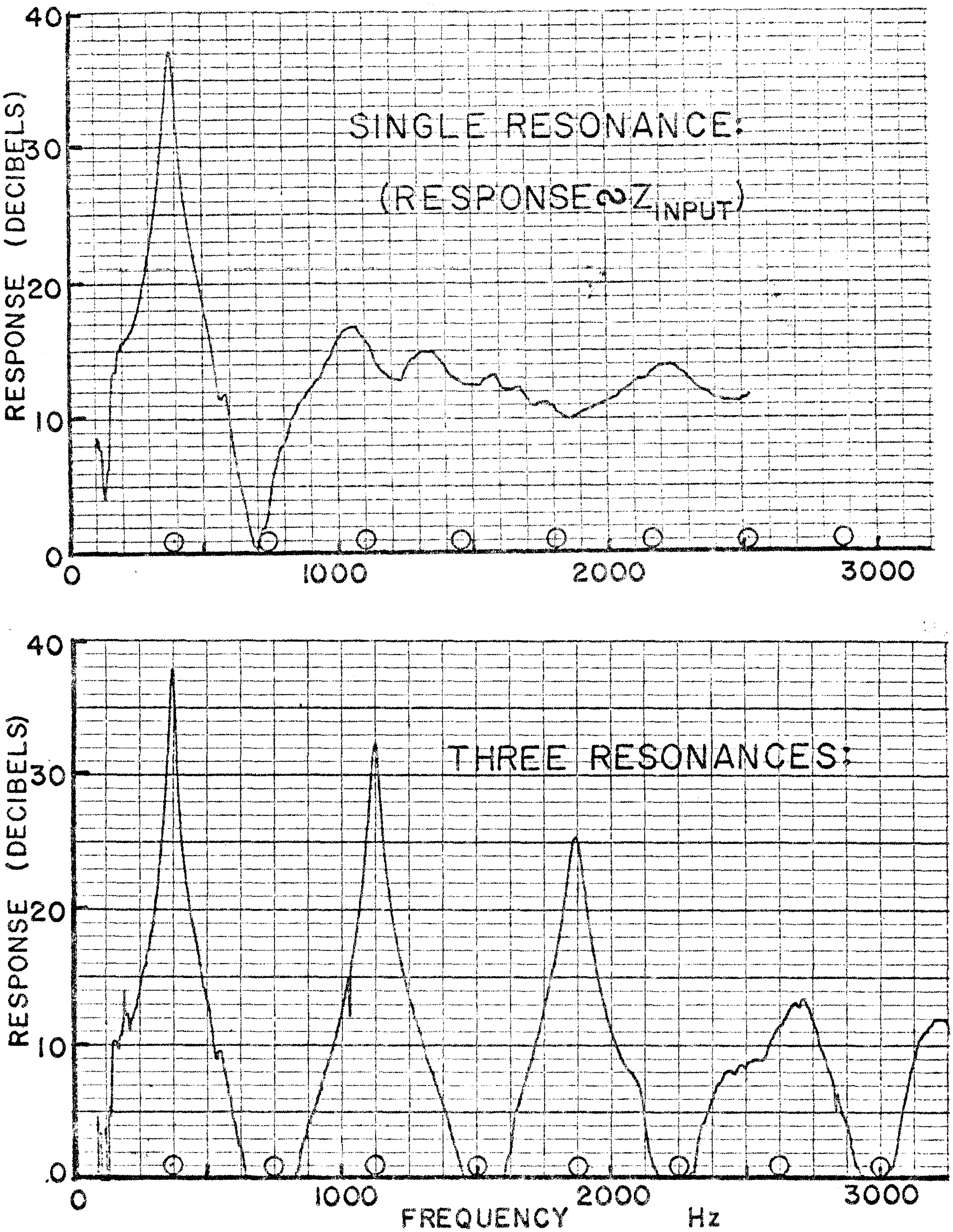
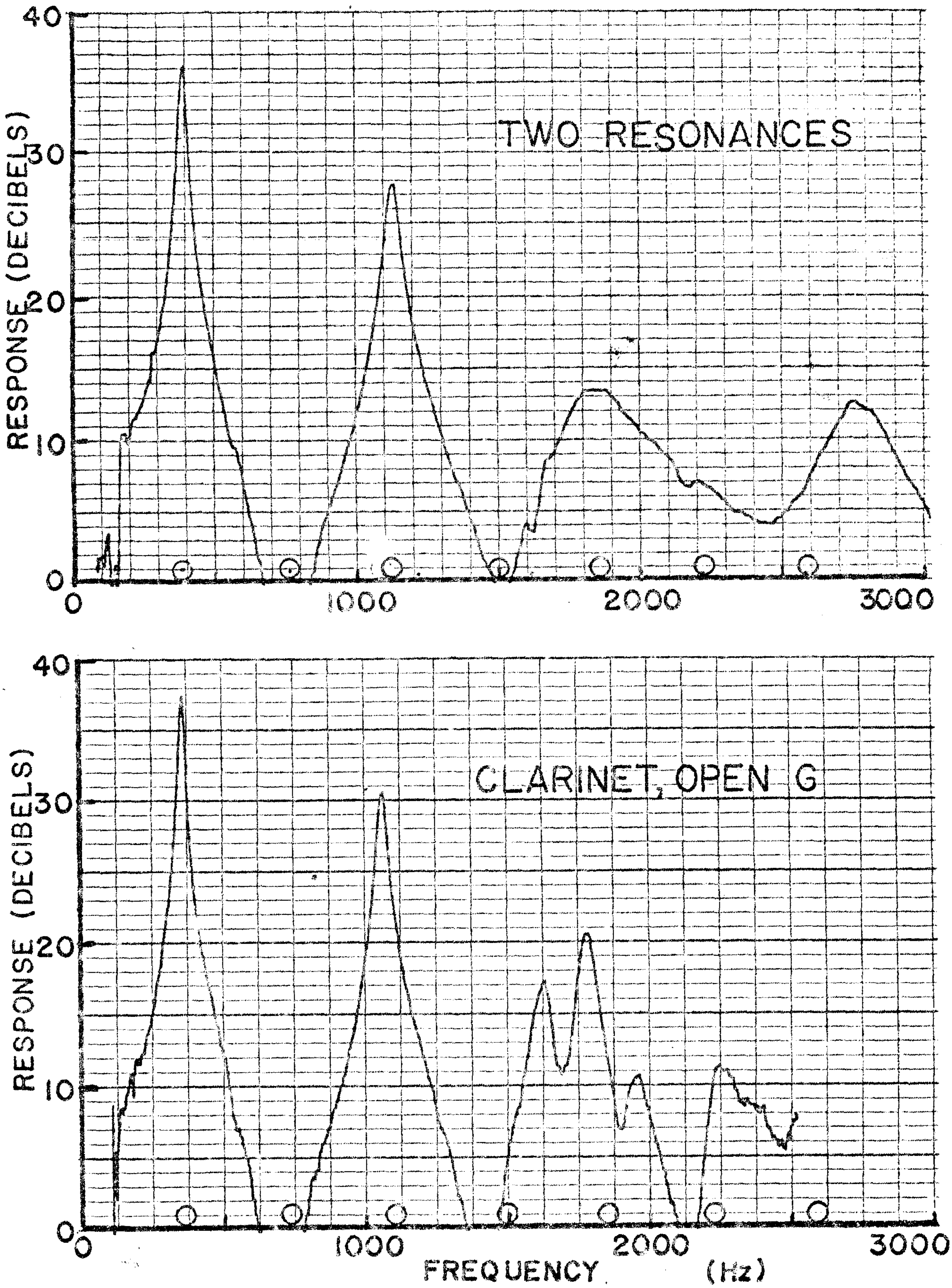

SINGLE RESONANCE:
(RESPONSE ∼ Z INPUT)
RESPONSE (DECIBELS)
0
10
20
30
40
0
1000
2000
3000
THREE RESONANCES:
RESPONSE (DECIBELS)
FREQUENCY Hz

FIGURE 3



TWO RESONANCES
RESPONSE (DECIBELS)
40
30
20
10
0
0
1000
2000
3000
CLARINET, OPEN G
RESPONSE (DECIBELS)
40
30
20
10
0
0
1000
2000
3000
FREQUENCY (Hz)

FIGURE 4