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Calculating the Spectrum of a Struck Piano String

A. H. Benade December 1983

Introductory Remarks

The question of what is the spectrum of a struck string comes up from time to time, and many piece-meal attacks have been made on it over the years. I have given an account of some of the basic phenomena in my "Fundamentals of Musical Acoustics" without presenting there much more than hints of the calculations and/or measurements that were carried out by me (either in years previous, or during the actual writing of the book) to support the account. There are of course references to the literature available at the time of writing.

For various reasons I am impelled to sketch out here at least the outline of a framework upon which one can build a coherent picture of what takes place. No attempt is made to present a complete picture. But I will assert that the formulation is one which may readily be adapted to the case of broad and soft hammers. No claim is made for anything new, only that many things which I and others have known about for many years can be made to come out in a consistent fashion upon the basis of a first-principles calculation.

The spectrum of a vibrating piano string is inexorably limited at the high frequency end by hammer softness effects and breadth effects that are outlined in FMA, Chapter 8 sections 2,3,4,5. At the time of writing this part of the book I worked out the underlying behavior along the following lines: The hammer blow is a force $f(x)\cdot g(t)$ that is distributed in time and space along the string. The excited spectrum is therefore band limited both by the upper temporal frequency limit of the Fourier transform $\phi(\omega) = \int g(t)\exp(-j\omega t)dt$ and by the spatial frequency limit of the transform $\phi(k) = \int f(x)\exp(-jkx)dx$. The relation $k = \omega/c$ for all disturbances on the string assures that both limitations govern what is going on.

The overall dynamics of the struck string is summarized in FMA Chapter 17 sec. 4, although the discussion of the clavichord in sec. 1 of Chapter 18 and the numbered statements on page 356 are germane to a discussion of string behavior between the time of arrival of the hammer and when it rebounds. Here, in a nutshell, is what goes on: If the hammer is pretty massive, and as long as it is in contact with the string, we have a pair of back-to-back clavichords. That is, there are (roughly speaking) two modal collections of vibrations, that may be said to belong to the two arbitrarily chosen string segments of length H and L (with $H + L = L_o$ being the overall string length). In this simplified initial view, we recognize that the modal frequencies on the two sides are of the type $f_n \approx nc/2H$ and $f_m \approx mc/2L$. The two sets of frequencies are not necessarily in simple relation to one another, since the L/H is not required to be integer. When the hammer separates from the string, we must start the calculations all over, where the subsequent configuration of the string $F(x,t)$ has for its initial shape $S_o(x)$ a composite of what was on the two string segments at the instant when the hammer left. That is

$$S_o(x) = \begin{pmatrix}\text{Shape on segment H,}\\ \text{used for } 0 < x < H\end{pmatrix} + \begin{pmatrix}\text{Shape on segment L,}\\ \text{used for } H \leq x \leq L_o\end{pmatrix}$$

There is of course a related velocity shape $V_o(x)$ based similarly on the segment velocity shapes. We next can use the usual Fourier expansions belonging to the complete, unmodified, string to deduce the motion that follows from these initial conditions.

In what follows we have: (A) A review of the vibrational properties of a nonuniform and/or discrete (or mixed), one dimensional chain, to lay the groundwork for what follows. This outline is closely based on some notes I use in our sophomore "Waves" course for physics majors.

(B) We calculate the normal modes of the composite string-plus-hammer system and set down their orthogonality relations. These are exact calculations, assuming no elasticity in the hammer contact. The method easily generalizes to include such things. See the "Digression" on page 332 of FMA for a carefully worded discussion of when it is appropriate to talk as though the string segments had their own modes.

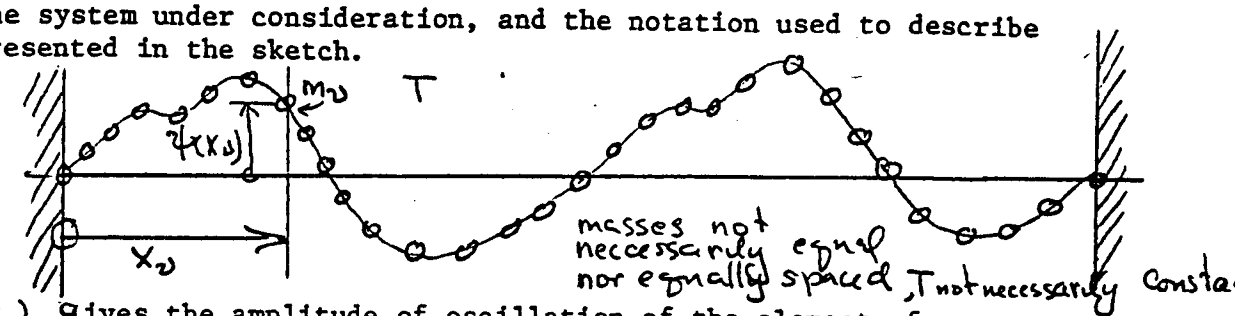
(C) Next comes an actual outline of what must be done to; (a) Calculate the motion during the hammer contact time, (b) The duration of the contact time. (c) The motion of the string after the hammer leaves it.

In closing this set of introductory remarks I should offer a warning and an apology. There may be "glitches" of detail and notation in these notes, because they are the work of a single busy afternoon, and were not really checked through afterward.

Some Properties of the Vibrational Modes of a 1-Dimensional Lumped/Courteous Chain

A. H. Benade Dec. 1983

The system under consideration, and the notation used to describe it is presented in the sketch.



Here $\psi(x_\nu)$ gives the amplitude of oscillation of the element of mass m_ν whose "address" along the chain is x_ν. The sequence of x_ν can be discrete or continuous. In the latter case $m_\nu = \mu(x_\nu)dx$, the mass of a short piece of chain having length dx and local linear mass density $\mu(x_\nu)$.

Newton's second law applied to the νth mass may now be written (for sinusoidal motion at the frequency ω)

$$(1) \qquad -\omega^2 m_\nu \Psi(x_\nu) = \left[\text{Tension} \times \text{Slope}\right]_{\text{right}} - \left[\text{Tension} \times \text{Slope}\right]_{\text{left}}$$

Note that the tension need not be constant on the two sides of the mass in question ["Tension" is a special case of any sort of inter-mass elastic coefficient]. For the case of continuously-distributed mass $\mu(x)$ and tension $T(x)$ we can write Eq. 1 explicitly in the form

$$(2) \qquad -\omega^2 \mu(x)\Psi(x) = \frac{\partial}{\partial x}\left[T(x)\frac{\partial \Psi}{\partial x}\right]$$

The discrete version is closely similar.

For any given sort of (essentially nondissipative) boundary conditions, we get a discrete set of Ψ_n's (each with its own modal frequency ω_n) which are orthogonal with a weight function determined by the mass distribution. That is, for two of these eigenfunctions Ψ_n and Ψ_p we have for the discrete case.

$$(3a) \qquad \sum_\nu \left[m_\nu \Psi_n(x_\nu)\Psi_p(x_\nu)\right] = \begin{cases} M_n & \text{if } p = n \\ 0 & \text{if } p \neq n \end{cases}$$

For the continuous version we get

$$(3b) \qquad \int \left[\mu(x)dx\right]\Psi_n(x)\Psi_p(x) = \begin{cases} M_n & \text{if } p = n \\ 0 & \text{if } p \neq n \end{cases}$$

It is customary to call M_n the nth modal mass, for good reasons which need not concern us here. Note that the continuity or discreteness of T does not appear explicitly in the orthogonality relation.

In a composite system, partly discrete and partly continuous, Eq. 3 must of course be written as the combination of the sum over all of the discrete $\psi_n(x_\nu)\psi_p(x_\nu)m_\nu$ terms and the integral over the part in which everything is continuous. After all ψ_n (any given "address") merely tells about the motion of the piece of mass that lives at this address, and cares not about the details of how the mass is distributed.

The orthogonality property of Eq. 3 allows us to represent any possible chain motion F(x,t) as a sum over a suitably weighted collection of ψ's. Thus, for free vibration we can write

$$(4) \quad F(x_\nu,t) = \sum_n \Psi_n(x_\nu)\left[A_n \cos\omega_n t + B_n \sin\omega_n t\right]$$

The A_n and B_n coefficients are obtained, for example, from the initial conditions we impose of $F(x_\nu,t)$. Let us write

$$S_0(x_\nu) = F(x_\nu,0) \rightarrow \text{Initial displacement distrib}$$

$$V_0(x_\nu) = \dot{F}(x_\nu,0) \rightarrow \text{Initial velocity distrib}$$

Then the nth amplitude coefficients for a discrete mass chain come out to be

$$(5a) \quad A_n = \frac{1}{M_n}\sum_\nu m_\nu S_0(x_\nu)\,\Psi_n(x_\nu)$$

$$(5b) \quad B_n = \frac{1}{M_n\omega_n}\sum m_\nu V_0(x_\nu)\,\Psi_n(x_\nu)$$

The continuous-chain, and mixed continuous/discrete versions of these equations are pretty obvious.

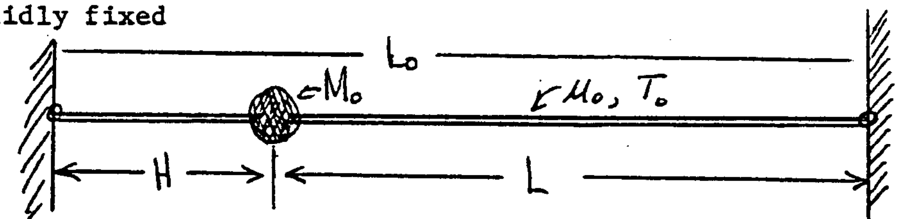
Normal Modes of a Piano String with Large Hammer Mass Attached

A. H. Benade 3 December 1983

For a short period of time after the hammer strikes a piano string, the two move together. In order to find the string motion after the hammer has rebounded, we must calculate what is the motion of the composite string/hammer system at the instant of separation. This can best be done upon the basis of a normal modes expansion. The present note outlines the determination of the modal frequencies and wave functions.

A. Calculation of Mode Frequencies

Consider a uniform string of length L_o, having uniform linear mass density μ_o and uniform tension T_o. We attach a compact mass M_o to the string at a distance H from the left end, as sketched. The string ends are solidly fixed



We recall that the wave velocity c on the string is $\sqrt{T_o/\mu_o}$ while its (force/velocity) wave impedance R_o is $\sqrt{T_o\mu_o}$. If we attempt to drive the string sinusoidally at the frequency ω, the required force is the sum of forces needed to drive the following three items: The left hand part of the string (of length H), the right hand part (of length L) and the mass M_o. All three of these are fastened together so that they have a common velocity v. From the definition of impedance, we have then

$$\text{Net drive force} = Z_{mass} \times v + Z_{left\ string} \times v + Z_{right\ string} \times v$$

From this, then

$$(1) \qquad Z_{\substack{net\ seen\\ by\ driver}} \left(= \frac{\text{Net force}}{\text{Velocity}}\right) = Z_{mass} + Z_{\substack{left\\ string}} + Z_{\substack{right\\ string}}$$

Recall, this is the impedance seen by the driver when it makes the system vibrate (willingly or unwillingly) at the imposed frequency. The normal mode frequencies ω_n are those at which the system will run without need of external constraint.

That is, $F_{drive}(\omega_n) = 0$ and so we get an eigenvalue equation.

$$(2a) \qquad Z_{mass}(\omega_n) + Z_{left\ string}(\omega_n) + Z_{right\ string}(\omega_n) = 0$$

or explicitly

$$(2b) \qquad j\omega_n M_o + (R_o/j)[\cot k_n H + \cot k_n L] = 0$$

Tidy it up a little, using $\omega_n = k_n c$.

$$(3) \qquad \cot k_n H + \cot k_n L = k_n \left(\frac{M_o c}{R_o}\right)$$

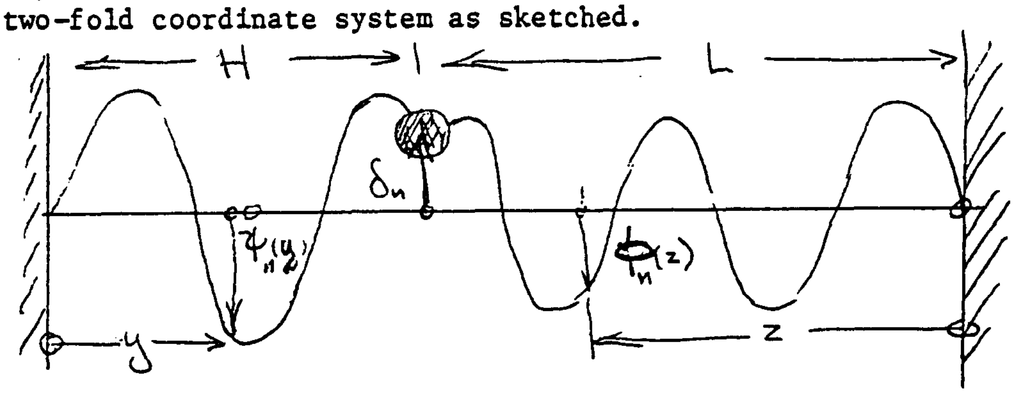
This is a very easy equation to solve for the successive k_n's ... merely set a computer to work. As a quickie look at what will happen, we note that for a sufficiently large M_o/R_o ratio, the solutions are those which make

$$(4) \qquad k_n \cong (\pi/H) \times \text{integer} \quad \text{or} \quad k_n \cong \left(\frac{\pi}{L}\right) \times \text{Integer}$$

These are of course the modal frequencies of the left and right hand string segments if they were to run in isolation. Note that <u>both sides</u> of the composite system vibrate at each of these frequencies regardless of its affinity for one or another segment of the string. We will find at the end of part B below however that the right hand string segment has only a small disturbance at frequencies that satisfy the $k_n \approx (\pi/H)$ x integer requirement, while the left hand part hardly moves in modes for which $k_n \approx (\pi/L)$ x integer.

B. Calculation of the Mode Shapes

Once we have found the values of the various k_n's (and thence the ω_n's if we want) using the methods outlined above, we are in a position to deduce the wave functions themselves. In order to get the string anchorage boundary conditions into the equations in the most direct fashion, we set up a two-fold coordinate system as sketched.



Here, the left and right strings have their displacements ψ_n and ϕ_n defined in terms of the (known) k_n and a sine function measured in from the "nearest") end via the coordinates y and z. The displacement of the hammer mass M_o is denoted by δ_n. That is

$$(5) \qquad \begin{aligned} \Psi_n(x)_{\text{left part of string}} &\equiv \alpha_n \sin k_n y &\equiv \psi_n(y) \\ \Psi_n(x)_{\text{right part of string}} &\equiv \beta_n \sin k_n z &\equiv \phi_n(z) \end{aligned}$$

We find it convenient to specify everything in terms of the motion of the mass M_o, so that

$$(6) \qquad \alpha_n = \delta_n / \sin k_n H \quad \text{and} \quad \beta_n \cdot \delta_n / \sin k_n L$$

To summarize: The wavefunctions $\Psi_n(x_\nu)$ that fits the string end boundary conditions, and the junction requirements at M_o have now been found. For convenience, we have chosen to use different means for specifying the "addresses" of the various mass particles making the system.

We can remark here that if M_o/R_o is large, then modes for which $k \approx (\pi/H) \times$ integer will have a standing wave amplitude on the right hand portion of the string smaller by a factor $(\mu_o H/M_o)(1/\pi \times$ that integer$) \ll 1$, than that on the left (provided this k is about midway between the nearest k's "belonging" to the right hand side of the string). A similar factor with L replacing H applies to the amplitude on the left when $k \approx (\pi/L) \times$ integer.

C. Orthogonality Property of the Modal Wave Function

We write out the orthogonality integral explicitly. Recall it is a three-segment summation

$$(7) \quad \int_0^H [\mu_o dy]\left[\delta_n \frac{\sin k_n y}{\sin k_n H}\right]\left[\delta_p \frac{\sin k_p y}{\sin k_p H}\right] + M_o \delta_n \delta_p + \int_0^L [\mu_o dz]\left[\delta_n \frac{\sin k_n z}{\sin k_n L}\right]\left[\delta_p \frac{\sin k_p z}{\sin k_p L}\right] = \begin{cases} M_n & \text{if } p = n \\ 0 & \text{if } p \neq n \end{cases}$$

Note, the $(\sin k_n y)(\sin k_p y)$ product integrated over H, and its z-coordinate cognate integrated over L are not orthogonal in and of themselves, except in the $(M_o/R_o) \to \infty$ limit. It is an interesting exercise to understand why both δ's here can be made equal in magnitude to one another, and even made equal to unity (as we shall do)

It is always a good idea to grind out Eq. 7 to check the correctness of the wave functions to which it is presumed to apply.

Vibrational Shape of A Struck String

A. H. Benade 3 December 1983

When a hammer of mass M_o strikes a string of length L_o, a distance H from its left end, we need to consider the motion in two stages. A. The epoch during which the hammer is in contact with the string. B. The subsequent times after the hammer has rebounded from the string.

A. Motion while String and Hammer are in Contact

Suppose that at t = 0 the hammer arrives at the previously stationary string with a velocity v_o. In the notation of the A-set of notes dated 3 December 1983 on the properties of a one-dimensional chain, we have the initial shape $S_o(x)$ of the string/hammer system to be zero everywhere, while the initial velocity distribution is $V_o(x) = v_o \delta(x-H)$. If as in Eq. 4 of the previous notes, the vibrational shape is F(x,t) is written in terms of a sum over vibrational modes,

$$(1) \qquad F(x,t) = \sum \Psi_n(x)\left[A_n \cos\omega_n t + B_n \sin\omega_n t\right]$$

Then Eq. 5a from before shows that all the A_n's are identically zero, while reference to the explicit forms for the ψ's given in the B set of notes from 3 December on the modes of a string plus hammer leads us to the following formula for the B_n's.

$$(2) \qquad B_n = \left(\frac{v_o}{M_n\omega_n}\right)\left\{\frac{\mu_o \cancel{\sin k_n H}}{\cancel{\sin k_n H}} + M_o + \frac{\mu_o \cancel{\sin k_n L}}{\cancel{\sin k_n L}}\right\}$$

$$= \left(\frac{v_o}{M_n\omega_n}\right)\left(2\mu_o + M_o\right\}$$

Here ω_n is the n'th modal frequency of the composite system, k_n the corresponding wave number $\omega_n k$ on the string segments, M_o is the linear mass density of the string, and M_n is the modal mass as defined on Eq. 3 of the A set of notes and written out explicitly for the string/mass system in Eq. 7 of the B set of notes.

B. Duration of the Contact Time Between String and Hammer

Because the hammer is not glued to the string, we need only find the time at which the acceleration of hammer (and thence the force exerted to cause its motion) changes sign in the motion calculated in the preceding part of these notes. That is, we find the earliest time τ for which the following relation holds

$$(3a) \qquad \left[\frac{d^2}{dt^2} F(x,t)\right]_{\substack{x=H \\ t=\tau}} = 0$$

That is

$$(3b) \qquad \sum \omega_n^2 \Psi_n(H)\, B_n \cos\omega_n\tau = 0$$

Once again, the computer can be asked to do a tedious but straightforward job in solving this.

C. Vibration Recipe of the Free String after Hammer Separation

For this case the displacement and velocity shapes at t=0 (measured from the instant of separation) are the t = τ shapes belonging to the composite string/hammer system at the instant of separation. The time τ being that found in part B of these present notes. Explicitly then, using the (y,z) notation of the composite system we have,

$$(4a) \qquad \underset{\text{of the string alone}}{S_0(x)} = F(x,\tau) \text{ of the composite system}$$

$$(4b) \qquad \underset{\text{of the string alone}}{V_0(x)} = \dot{F}(x,\tau) \text{ of the composite system}$$

where $x = y$ for $0 < x < H$, and $x = (L_0 - z)$ for $H < x < L_0$.

We wish to use these in a calculation of the A_p, B_p amplitudes belonging to the free-string disturbance $F_{\text{free string}}(x,t)$

$$(5) \qquad \underset{\text{free string}}{F(x,t)} = \sum_p \Psi_p(x)\left[A_p \cos\omega_p t + B_p \sin\omega_p t\right]$$

where

$$(6) \qquad \Psi_p(x) = \gamma \sin k_p x$$

and

$$(7) \qquad k_p = p\pi/L_0 \quad \text{with} \quad \omega_p = k_p c$$

Because $S_o(x)$ and $V_o(x)$ are written in terms of the segmental coordinates y and z, it is convenient here to rewrite the free string wave functions in the corresponding way

$$(8) \qquad \Psi_p(x) = \begin{cases} \gamma \sin k_p y \quad (\equiv \psi_p(y)) & \text{for } 0 \le x \le H \\ \gamma \sin k_p z \quad (\equiv \Phi_p(z)) & \text{for } H \le x \le L \end{cases}$$

with γ adjusted to make $\psi_p(H) = 1$. Notice that this segmental notation does not upset the continuity of ψ_p and its derivative across the point x = H from which the hammer has just departed.

Using the form of ψ_p given in Eq. 8 we then can calculate the A_p and B_p coefficients for Eq. 5 above via the following repetition of Eq. 5 from the A set of notes

see next page

$$(9a) \quad -A_p = \frac{1}{M_n}\left\{ \int_0^H (\mu_0 dy)\, F(y,\tau)\, \Psi_p(y) + \int_0^L (\mu_0 dz)\, F(z,\tau)\, \Phi_p(z) \right\}$$

composite string + mass

composite

Note that M_p for the complete string comes to be $(M_o L_o)/(2\sin^2 k_p H)$.